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A THEORETICAL INVESTIGATION OF PARTICLE TRAJECTORIES THROUGH A PRANDTL-MEYER EXPANSION FAN

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ABSTRACT

The equations of motion are developed and solved numerically for the trajectory of a spherical particle passing through a Prandtl-Meyer expansion fan. The effect of a change in γ and n is shown, where γ is the ratio of specific heats of the gas, and $n/2$ is the exponent in

$$\frac{\mu}{\mu_o} = \left(\frac{T}{T_o}\right)^{n/2} ,$$

the assumed viscosity temperature relationship.

It is demonstrated that for particles the order of a micron in diameter, slip flow will exist, and a proposed correction to the drag to account for this discontinuous nature of the flow is investigated.

The results are plotted showing particle trajectory profiles and the components of the relative velocity or slippage velocity of the particle for flow deflection angles up to 70 degrees.

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I. INTRODUCTION

Heterogeneous flow of a solid particle - gas mixture has been a topic of increased interest in recent years. The addition of light metals to solid rocket fuels has focused attention on the non-equilibrium nature of two-phase flows in expansion processes, and numerous papers have been written regarding this phenomenon.

Early studies of two-phase flows were involved with investigations of water droplet trajectories around a variety of body shapes in connection with icing of aircraft surfaces. More recently, heterogeneous nozzle flow has been examined. Gilbert, Davis, and Altman (reference 1) related rocket thrust losses to solid particle sizes and solved linearized nozzle equations assuming Stokes drag law. Kliegel (reference 2) treated simultaneous velocity and thermal lag along with the effect of particle lag on gas properties in a one-dimensional nozzle. Rannie (reference 3) showed that for most cases of interest, results of equal accuracy to numerical integration can be obtained through a perturbation analysis of temperature and velocity lags in a one-dimensional nozzle. Rannie also introduced slip flow corrections for drag and heat transfer. An excellent summary of recent advances in heterogeneous nozzle flow was presented at the American Rocket Society, Solid Propellant Rocket Conference in January 1962, by R. F. Hoglund (reference 4).

Several problems relating to supersonic flow of two-phase systems have also been examined. Carrier (reference 5) investigated a solution to particle flow through normal shocks, and oblique shocks were treated by Morgenthaler (reference 6).

The behavior of a single particle passing through a Prandtl-Meyer expansion furnishes an example of heterogeneous flow that can be analyzed in a straightforward manner, and serves to illustrate some of the problems and important parameters of two-phase flow in general. This paper presents the development of the equations of motion of small spherical particles flowing with a gas that experiences a Prandtl-Meyer expansion, and the numerical solution of these equations. The effects of the temperature dependence of viscosity, changes in the ratio of gas specific heats, γ, and slip flow correction to the drag coefficient were investigated by varying these factors in successive solutions. The results of this investigation have direct bearing on the exhaust of an underexpanded rocket nozzle as the flow expands around the lip of the nozzle.

II. DEVELOPMENT OF BASIC EQUATIONS

An ideal two-dimensional Prandtl-Meyer expansion about a sharp corner in a semi-infinite channel is the assumed model for this investigation. A cylindrical coordinate system with origin at the vertex of the expansion fan is used to describe the position of a spherical particle passing through the fan. Denoting the radial distance from the origin to the center of the particle as , and the angular displacement of the particle from the initial characteristic as (Fig. 1), the conservation of momentum equations can be written

$$\frac{d}{dt}(m\dot{r}) = \frac{m(r\dot{\theta})^2}{r} + f\,6\pi\sigma\mu\,(u_g - \dot{r}) \tag{1}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = f\,6\pi\sigma\mu\,(v_g - r\dot{\theta})\,r \quad . \tag{2}$$

It is assumed that the only force acting on the particle is a drag force which can be represented by Stokes drag law modified by the factor f . The radial and tangential components of the gas velocity are u_g and v_g respectively, and μ is the local value of the gas viscosity. The particle has mass m and radius σ .

It is convenient to rewrite the equations with θ the independent variable. Choosing $L = mr^2\dot{\theta}$, the angular momentum, and $1/r$ as the desired dependent variables, the time derivative can be written

$$\frac{d}{dt}(\quad) = \frac{L}{mr^2}\frac{d}{d\theta}(\quad) .$$

Equation 1 then becomes

$$\left(\frac{L}{mr^2}\right)\frac{d}{d\theta}\left[m\left(\frac{L}{mr^2}\right)\frac{dr}{d\theta}\right] = \frac{L^2}{mr^3} + f\,6\pi\sigma\mu u_g$$

$$- f\,6\pi\sigma\mu\left(\frac{L}{mr^2}\right)\frac{dr}{d\theta} \quad .$$

Putting $\dfrac{dr}{d\theta} = -r^2\dfrac{d(1/r)}{d\theta}$ and multiplying by mr^2

$$L^2\frac{d^2(1/r)}{d\theta^2} + L\frac{d(1/r)}{d\theta}\left(\frac{dL}{d\theta}\right) + \frac{L^2}{r}$$

$$+ f\,6\pi\sigma\mu m r^2 u_g + f\,6\pi\sigma\mu L r^2\frac{d(1/r)}{d\theta} = 0 \quad . \qquad (3)$$

Equation 2 is simply

$$\frac{dL}{d\theta} = f\,6\pi\sigma\mu\left(\frac{mr^3 v_g}{L} - r^2\right) \quad . \qquad (4)$$

Replacing $\dfrac{dL}{d\theta}$ in the second term of equation 3 by equation 4 we have

$$L^2\frac{d^2(1/r)}{d\theta} + \frac{L^2}{r} + f\,6\pi\sigma m r^3 v_g\frac{d(1/r)}{d\theta}$$

$$+ f\,6\pi\sigma m r^2 u_g = 0 \quad . \qquad (5)$$

Non-dimensional parameters are defined as

$$R = \frac{r}{\varsigma} \qquad \text{and} \qquad \Lambda = \frac{L}{mc\varsigma\lambda} \qquad (6)$$

where $\lambda = \sqrt{\dfrac{\gamma-1}{\gamma+1}}$, γ is the ratio of gas specific heats, and C is the theoretical maximum expansion velocity determined by the equation

$$C^2 = 2c_pT_o\left(1 + \frac{\gamma-1}{2}M_o^2\right) \quad . \qquad (7)$$

The characteristic length, ς , has the value

$$\varsigma = \frac{mc\lambda^2}{6\pi\sigma\mu_o} \quad . \qquad (8)$$

The subscript zero indicates initial conditions of the gas or particle as they enter the expansion fan.

Introducing equations 6 into equation 5, we obtain

$$\frac{L^2}{\varsigma}\left[\frac{d^2(1/R)}{d\theta^2}+\frac{1}{R}\right]+f\varsigma^2 6\pi\sigma\mu m R^2\left[R v_g \frac{d(1/R)}{d\theta}+u_g\right]=0 \quad ,$$

and dividing by $\varsigma^2 6\pi\sigma m\mu_o c$,

$$\Lambda^2\left[\frac{d^2(1/R)}{d\theta^2}+\frac{1}{R}\right]+f R^2\frac{\mu}{\mu_o}\left[R\frac{v_g}{c}\frac{d(1/R)}{d\theta}+\frac{u_g}{c}\right]=0 \quad . \tag{9}$$

Equation 4 can be written in terms of the dimensionless parameters as

$$mc\varsigma\lambda\frac{d\Lambda}{d\theta}=f\,6\pi\sigma\mu R^2\varsigma^2\left(\frac{R}{\lambda\Lambda}\frac{v_g}{c}-1\right)$$

or

$$\frac{d\Lambda}{d\theta}=fR^2\frac{\mu}{\mu_o}\left[\frac{R}{\Lambda}\frac{v_g}{c}-\lambda\right] \quad . \tag{10}$$

The conservation equations of frictionless flow yield $\frac{u_g}{c}$, $\frac{v_g}{c}$, and $\frac{T}{T_o}$ as functions of θ . The equations may be written in terms of our assumed coordinate system as

Continuity: $$\rho\left(u_g+\frac{d v_g}{d\theta}\right)+v_g\frac{d\rho}{d\theta}=0 \tag{11}$$

Momentum:

$$\frac{v_g}{r}\frac{du_g}{d\theta}-\frac{v_g^2}{r}=0 \tag{12}$$

$$\frac{v_g}{r}\left(u_g+\frac{dv_g}{d\theta}\right)=\frac{a^2}{\rho r}\frac{d\rho}{d\theta} \tag{13}$$

Energy: $$a^2 = \gamma R_g T = \frac{\gamma - 1}{2}\left(c^2 - u_g^2 - v_g^2\right) . \tag{14}$$

Equations 11 and 13 can be recombined to eliminate the term in the parentheses and show that

$$v_g = a . \tag{15}$$

Using this result in equation 14, v_g can be found as a function of u_g which can, in turn, be used in equation 12 to give

$$\frac{u_g}{c} = \sin(\lambda\theta + \alpha) ,$$

$$\frac{v_g}{c} = \lambda\cos(\lambda\theta + \alpha) . \tag{16}$$

The value of the constant α can be found by evaluating equations 16 at the initial conditions $\theta = 0$, $v_g = a_o$, and using equation 14 to give

$$\cos^2\alpha = \frac{1}{\lambda^2(M_o^2 - 1) + 1} . \tag{17}$$

The quantity $\frac{T}{T_o}$ can be found by applying equation 14 to show that

$$\frac{T}{T_o} = \frac{1 - \left(\frac{u_g}{c}\right)^2 - \left(\frac{v_g}{c}\right)^2}{1 - \left(\frac{u_g}{c}\right)_o^2 - \left(\frac{v_g}{c}\right)_o^2} ,$$

or by equations 16

$$\frac{T}{T_o} = \frac{\cos^2(\lambda\theta + \alpha)}{\cos^2\alpha} . \tag{18}$$

It is assumed that viscosity varies with temperature raised to a fractional power, hence

$$\frac{\mu}{\mu_o} = \left(\frac{T}{T_o}\right)^{\frac{n}{2}} , \tag{19}$$

or from equation 18

$$\frac{\mu}{\mu_0} = \frac{\cos^n(\lambda\theta + \alpha)}{\cos^n \alpha} \quad , \tag{20}$$

where n will normally be between 1.0 and 2.0.

The equationsof particle motion (9 and 10) can now be written entirely in terms of the dimensionless variables R , Λ , θ , the drag correction factor f , and the constants α and λ . Letting $(\ )'$ denote $\frac{d(\ )}{d\theta}$ they become

$$\Lambda^2\left[\left(\frac{1}{R}\right)'' + \frac{1}{R}\right] + f R^2 \frac{\cos^n(\lambda\theta+\alpha)}{\cos^n \alpha}\left[R\left(\frac{1}{R}\right)' \lambda \cos(\lambda\theta+\alpha) + \sin(\lambda\theta+\alpha)\right] = 0 \quad , \tag{21}$$

$$\Lambda' - f R^2 \frac{\cos^n(\lambda\theta+\alpha)}{\cos^n\alpha}\left[\frac{R}{\Lambda}\lambda \cos(\lambda\theta+\alpha) - \lambda\right] = 0 \quad . \tag{22}$$

A value of f for spheres in the subsonic slip flow regime has been proposed by W. D. Rannie (ref. 3) which, for relative Mach numbers less than unity and for Reynolds numbers less than about 0.5, takes the form

$$\frac{1}{f} = 1.0 + 4.05\sqrt{\gamma}\,\frac{M_s}{Re} \quad , \tag{23}$$

where M_s and Re are the relative Mach number and Reynolds number respectively.

$\frac{M_s}{Re}$, which is related to the ratio of molecular mean free path to the sphere diameter, can be evaluated as follows:

$$\frac{M_s}{R_e} = \frac{\mu_\circ}{2 a_\circ \rho_\circ \sigma} \left(\frac{\mu}{\mu_\circ}\right)\left(\frac{a_\circ}{a}\right)\left(\frac{\rho_\circ}{\rho}\right) . \tag{24}$$

From equations 15 and 16

$$\frac{a_\circ}{a} = \frac{\cos \alpha}{\cos(\lambda \theta + \alpha)} . \tag{25}$$

Isentropic relationship between density and temperature along with equation 19 gives

$$\frac{\rho}{\rho_\circ} = \left[\frac{\cos \alpha}{\cos(\lambda \theta + \alpha)}\right]^{\frac{2}{\gamma - 1}} . \tag{26}$$

Using equations 20, 25, and 26 in equation 24, we get

$$\frac{M_s}{R_e} = \frac{\mu_\circ}{2 \rho_\circ a_\circ \sigma} \left[\frac{\cos \alpha}{\cos(\lambda \theta + \alpha)}\right]^{\frac{1}{\lambda^2} - n} , \tag{27}$$

and equation 23 becomes

$$\frac{1}{f} = 1 + \frac{2.02 \sqrt{\gamma}\, \mu_\circ}{\rho_\circ a_\circ \sigma} \left[\frac{\cos \alpha}{\cos(\lambda \theta + \alpha)}\right]^{\frac{1}{\lambda^2} - n} \tag{28}$$

The drag correction factor cannot be written in terms of the dimensionless similarity parameters R and Λ , but is dependent on the particle size and initial conditions.

It is convenient to relate the trajectory of the particle to that of a particle having zero relative velocity, i.e., moving along a gas streamline. Designating this path as $r_g = r_g(\theta)$, we can write

$$\frac{d r_g}{d t} = u_g = \frac{d r_g}{d \theta} \cdot \frac{d \theta}{d t} ,$$

and since $\dfrac{d\theta}{dt} = \dfrac{v_g}{r_g}$ we have

$$\frac{dr_g}{d\theta} = \frac{r_g u_g}{v_g} \quad . \tag{29}$$

Integrating this equation we obtain

$$\frac{r_g}{r_o} = \frac{R_g}{R_o} = \left(\frac{\cos\alpha}{\cos(\lambda\theta+\alpha)}\right)^{\frac{1}{\lambda^2}} . \tag{30}$$

From the definitions of R and Λ it follows that

$$\frac{v_g\, R_g}{v_o\, R_o} = \frac{\Lambda_g}{\Lambda_o} \quad ,$$

which, when combined with equations 16 and 30, gives

$$\frac{\Lambda_g}{\Lambda_o} = \left(\frac{\cos\alpha}{\cos(\lambda\theta+\alpha)}\right)^{\frac{1}{\lambda^2}-1} . \tag{31}$$

Two new dimensionless parameters are defined

$$X = \frac{R_g}{R} \quad ,$$

$$Y = \frac{\Lambda}{\Lambda_g} \quad , \tag{32}$$

from which it follows that

$$\frac{1}{R} = \frac{X}{R_o}\left[\frac{\cos(\lambda\theta+\alpha)}{\cos\alpha}\right]^{\frac{1}{\lambda^2}} , \tag{33}$$

$$\left(\frac{1}{R}\right)' = \frac{1}{R_o}\left[\frac{\cos(\lambda\theta+\alpha)}{\cos\alpha}\right]^{\frac{1}{\lambda^2}}\left[X' - \frac{X}{\lambda}\tan(\lambda\theta+\alpha)\right] , \tag{34}$$

$$\left(\frac{1}{R}\right)'' = \frac{1}{R_o}\left[\frac{\cos(\lambda\theta+\alpha)}{\cos\alpha}\right]^{\frac{1}{\lambda^2}}\left[X'' - \frac{2X'}{\lambda}\tan(\lambda\theta+\alpha)\right.$$

$$\left. + X\left\{\left(\frac{1}{\lambda^2}-1\right)\tan^2(\lambda\theta+\alpha) - 1\right\}\right] \quad , \tag{35}$$

$$\Lambda = Y\Lambda_o\left[\frac{\cos\alpha}{\cos(\lambda\theta+\alpha)}\right]^{\frac{1}{\lambda^2}-1} \quad , \tag{36}$$

$$\Lambda' = \Lambda_o\left[\frac{\cos\alpha}{\cos(\lambda\theta+\alpha)}\right]^{\frac{1}{\lambda^2}-1}\left[Y' + Y\left(\frac{1}{\lambda}-\lambda\right)\tan(\lambda\theta+\alpha)\right] \quad . \tag{37}$$

Also from equations 6 and 16

$$\Lambda_o = R_o\cos\alpha \, . \tag{38}$$

Writing equations 21 and 22 in terms of X , Y parameters using equations 33 through 38, we obtain the result

$$X'' + X\left(\frac{1}{\lambda^2}-1\right)\tan^2(\lambda\theta+\alpha) - \frac{2X'}{\lambda}\tan(\lambda\theta+\alpha)$$

$$+ f\lambda R_o\left(\frac{X'}{X^3Y^2}\right)\frac{(\cos\alpha)^{\frac{1}{\lambda^2}-n}}{(\cos(\lambda\theta+\alpha))^{\frac{1}{\lambda^2}-n+1}} \quad , \tag{39}$$

$$Y' + Y\left(\frac{1}{\lambda} - \lambda\right)\tan(\lambda\theta + \alpha)$$

$$- f\lambda R_0\left(\frac{1 - XY}{X^3 Y}\right)\frac{(\cos\alpha)^{\frac{1}{\lambda^2} - n}}{\left(\cos(\lambda\theta + \alpha)\right)^{\frac{1}{\lambda^2} - n + 1}} = 0 \; . \tag{40}$$

It is assumed that the particle is moving with no relative velocity prior to reaching the first characteristic line of the expansion. The initial conditions are then

$$\begin{aligned} X(0) &= 1.0 \; , \\ Y(0) &= 1.0 \; , \\ X'(0) &= 0 \; . \end{aligned} \tag{41}$$

Equations 28, 39, 40, and 41 constitute the complete statement of the problem.

III. AUXILIARY EQUATIONS

In order to specify the trajectory, the velocity as well as the position of the particle should be known. The important parameters in terms of X and Y are then

$$\frac{v}{v_g} = \frac{\Lambda}{R} \cdot \frac{R_g}{\Lambda_g} = XY \quad , \tag{42}$$

$$\frac{R}{R_o} = \frac{R}{R_s} \cdot \frac{R_s}{R_o} = \frac{1}{X}\left[\frac{\cos\alpha}{\cos(\lambda\theta+\alpha)}\right]^{\frac{1}{\lambda^2}} \quad , \tag{43}$$

$$\frac{u}{u_g} = \frac{1}{u_g}\frac{dr}{dt} = \frac{v}{v_g} \cdot \frac{v_g}{u_g} \cdot \frac{1}{R} \cdot \frac{dR}{d\theta} \quad .$$

which, by 33, 34, and 16, is

$$\frac{u}{u_g} = XY\left[1 - \frac{X'}{X}\lambda\cot(\lambda\theta+\alpha)\right] \quad . \tag{44}$$

The relative velocity or slip velocity of the particle relative to the absolute gas velocity is simply

$$\frac{v_s}{v_g} = \frac{v_g - v}{v_g} \quad ,$$

$$\frac{u_s}{u_g} = \frac{u_g - u}{u_g} \quad . \tag{45}$$

The relative Mach number is then

$$M_s = \sqrt{\left(\frac{v_s}{a}\right)^2 + \left(\frac{u_s}{a}\right)^2} \quad ,$$

or

$$M_s = \frac{v_s}{v_g} \sqrt{1 + \left(\frac{u_s}{v_s}\right)^2} \quad . \tag{46}$$

IV. SOLUTION OF EQUATIONS

Since equations 39 and 40 are non-linear, a numerical solution is indicated for arbitrary values of R_o. For either $R_o >> 1.0$ or $R_o << 1.0$, the possibility of a small perturbation solution exists; however, this possibility was not investigated.

Equations 21 and 22 were first solved utilizing a desk calculator and a simple iteration procedure where

$$\left(\frac{1}{R}\right)_2 = \left(\frac{1}{R}\right)_1 + \frac{1}{2}\left[\left(\frac{1}{R}\right)'_{\substack{R_1\\ \Lambda_1}} + \left(\frac{1}{R}\right)''_{\substack{R_1\\ \Lambda_1}} \Delta\theta\right]\Delta\theta ,$$

$$\left(\frac{1}{R}\right)'_2 = \left(\frac{1}{R}\right)'_1 + \left(\frac{1}{R}\right)''_1 \Delta\theta , \tag{47}$$

$$\Lambda_2 = \Lambda_1 + \frac{1}{2}\left[\Lambda'_{\substack{(R_1\\ \Lambda_1)}} + \Lambda'_{\substack{(R_2\\ \Lambda = \Lambda_1 + \Lambda'_1 \Delta\theta)}}\right]\Delta\theta .$$

For these calculations it was assumed that $f = 1.0$ (Stokes drag law applies), $\gamma = 1.4$, $n = 2.0$, and $M_o = \sqrt{2}$. Various values of R_o between 0.1 and 100 were investigated; however, it was found that the iteration interval, $\Delta\theta$, had to be decreased as R increased to keep the process convergent. For R_o very small, the iteration interval had to be made very small again to get meaningful result (proper divergence from the zero deflection line).

To eliminate much of this difficulty, equations 39 and 40 were developed. A similar iteration procedure to that of equations 47 was initially used to investigate the new equations, and it was found that an iteration interval of 0.01 radians gave satisfactory results for R_o between 0.01 and 100.

Equations 39 and 40 appeared to be the more satisfactory of the two sets of equations for programming on a digital computer, since the starting solutions of numerical methods are closely related to the simple iteration procedure.

A Fortran statement for the solution of equations 39 and 40 was written utilizing a subroutine prepared by Mr. K. H. Redner of Computer Sciences Corporation. The subroutine applies the method of Runge-Kutta-Gills as a starting solution, and the Adams-Moulton predictor - corrector formulas as a continuing solution. The facilities of the Computer Center at the California Institute of Technology were used in writing the program, and the IBM 7090 computer at the Jet Propulsion Laboratory was used for the actual solution. A Fortran statement is included as Appendix A.

To investigate the general effect of changes in γ and n, Stokes drag law was assumed to apply, and solutions were obtained for six values of R_o between 0.01 and 50 for three sets of assumed constants.

1) $M_o = \sqrt{2}$, $\gamma = 1.4$, $n = 2.0$,

2) $M_o = \sqrt{2}$, $\gamma = 1.4$, $n = 1.0$,

3) $M_o = \sqrt{2}$, $\gamma = 1.2$, $n = 2.0$.

The exhaust conditions of a typical large rocket were assumed as initial conditions for the investigation of the effect of allowing f to vary according to equation 28. The initial conditions chosen were:

$$M_o = 3.0 \ ,$$

$$\mu_o = 9.55 \times 10^{-7} \ \frac{lbf - sec}{ft^2} \ ,$$

$$\rho_o = 0.0138 \ \frac{lbm}{ft^3} \ ,$$

$$a_o = 3{,}100 \ fps \ (T_o = 2820°R \ ,$$

$$\gamma = 1.25 \ ,$$

$$n = 1.0 \ .$$

Solutions were obtained for R_o of 0.1, 1.0, and 50 for two particle sizes, $\sigma = \frac{1}{2}$ micron and $\sigma = 2.0$ microns.

V. RESULTS AND DISCUSSION

The results of this investigation are plotted on Figs. 3 through 10 for 0° to 70° of flow deflection angle. Flow deflection angle, δ, was chosen as a more meaningful parameter than θ for plotting and analyzing the results since it is determined solely by the physical boundaries of the flow. The fan angle and deflection angle can be related by the usual Prandtl-Meyer functions.

Figure 2 is a graphical representation of equation 6 giving R_o as a function of r_o and σ for particles having a density of 210 lb/ft^3 (as for Al_2O_3). Initial gas characteristics plotted are for air at 100°F and 1000°F and for the assumed rocket exhaust conditions given in the previous section. R_o is not greatly affected by a change in initial gas conditions, but is primarily a function of r_o and σ. For particles between 1.0 μ and 4.0 μ in diameter and initially displaced by several cm. from the flow boundary (as in the case of a rocket exhaust), R_o will normally be greater than 10.

The data plotted in Figs. 3, 4, and 5 is based on the assumptions that $M_o = \sqrt{2}$, $\gamma = 1.4$, $n = 2.0$, and Stokes drag law applies with the factor $f = 1.0$. For these conditions, particles with $R_o > 50$ will have little deviation from the streamline, and slip velocities will be low. For $R_o < 0.01$ there will be large tangential slip and very little deviation from its original flow direction through the first 25° of gas deflection. Radial slip is small compared to tangential slip, being a maximum of .164 for $R_o = 0.01$ at about 25° flow deflection. Near this point the maximum relative Mach

number is .993 and maximum Reynolds number for a 4.0 μ particle is 8.38. For $R_0 = 10$, the maximum radial slip is less than one per cent while the maximum Mach number is only 0.151 and maximum Reynolds number for 4.0 μ particle is 2.02. The Mach numbers and Reynolds number both drop sharply on either side of the maximum values.

The assumed viscosity - temperature function, equation 19

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{\frac{n}{2}} , \tag{19}$$

is a well-established approximation that applies over a limited temperature range when n is assigned the constant value appropriate to this range. By allowing n to take its expected limiting values, its effect on the solution can be observed. Figures 6 and 7 show curves for which Stokes drag, $\gamma = 1.4$, and $M_0 = \sqrt{2}$ where assumed for $n = 1.0$ and $n = 2.0$. A small change in trajectory path and tangential slip velocity and a negligible change in radial slip velocity were observed. Thus, little error is introduced by the selection of an average value of n over an extended temperature range.

A change in γ from 1.4 to 1.2 does have a fairly large effect on the trajectory path which is due principally to the change in the gas streamline. The tangential slip velocity is increased also, which tends to increase the path deviation from the streamline, but this is a small effect. Radial slip velocity is not appreciably affected.

The drag correction factor, f, which allows a departure from Stokes drag law to compensate for slip flow or free molecular flow effects, was proposed by Rannie (ref. 3) as an analogue to a similar

development for heat transfer coefficients. Although it has not been verified experimentally, it does give proper results for the extreme values of $\frac{M_s}{R_e} >> 1.0$ and $\frac{M_s}{R_e} << 1.0$, and should give a good approximation for intermediate values. The factor $\frac{M_s}{R_e}$ can be shown to be (ref. 7)

$$\frac{M_s}{R_e} = 0.499\sqrt{\frac{8}{\pi \gamma}}\,\frac{\ell}{2\sigma} \; ,$$

where ℓ is the mean free molecular path of the gas. Thus, if $\frac{M_s}{R_e} >> 1.0$, free molecular flow exists, and if $\frac{M_s}{R_e} << 1.0$ continuum flow exists. The area between these limits is the slip flow regime.

From equation 27 we have, for $\gamma = 1.2$ and $n = 1.0$,

$$\frac{M_s}{R_e} = k\left(\frac{\cos \alpha}{\cos(\lambda\theta + \alpha)}\right)^{10} . \qquad (48)$$

For air or rocket exhaust gases at moderate temperatures, and for particles the order of a micron, k will lie between 0.03 and 0.3 . Thus, a particle of this general size, passing through a Prandtl-Meyer expansion, will start in the slip flow regime and pass into free molecular flow at large expansion angles as the enclosed cosine function in equation 48 becomes large compared to unity.

Figures 8, 9, and 10 show the effect of f on the trajectory paths and slip velocities for 1.0 micron and 4.0 micron diameter particles. The deviation from the uncorrected Stokes drag law trajectories is large for particles as small as 4.0 micron diameter,

and increases markedly as the particle size is reduced to 1.0 micron. In the case of a 1.0 micron diameter particle, an $R_o = 50$ does not closely follow the streamline as it did for Stokes drag, and $R_o = 0.1$ now shows little deflection from its original flow direction. For $R_o = 0.1$ a maximum relative Mach number of 0.924 and maximum Reynolds number of 6.73 is reached for a 4.0 micron particle. This compares to maximums of 0.853 and 6.51 respectively for Stokes drag.

The maximum values of Reynolds numbers obtained are greater than the limits set on equation 23 for the determination of f. A Reynolds number of 8.0 would change the equation to

$$\frac{1}{f} = 0.63 + 4.05\sqrt{\gamma}\,\frac{M_s}{R_e},$$

(ref. 3), which would make a 23 per cent error in f for a Mach number of 0.95. While this is an appreciable error, it is effective over a small portion of the total expansion and diminishes rapidly on both sides of maximum R_e, so it should not greatly affect the overall results.

VI. CONCLUDING REMARKS

In the flow regime where Stokes drag law applies, the trajectories of spherical particles passing through a Prandtl-Meyer expansion fan can be specified by a single dimensionless parameter, R_o, which is a function of initial gas conditions, initial radius, and particle diameter. If the Reynolds number is greater than about 0.5, the trajectories are also a function of Re. When the mean free path of the gas molecules is of the same order or large compared to the particle diameter, Stokes drag law must be modified to correct for the discontinuous nature of the flow, and the trajectories then become functions of three parameters, R_o, Re, and M_s/Re.

It was found that the temperature dependence of viscosity need not be precisely known to ensure satisfactory results, but changes in the specific heat ratio, γ, do appreciably affect the trajectory paths.

The radial velocity lags are in general very small compared with the tangential velocity lags, and can, in many cases, be neglected completely.

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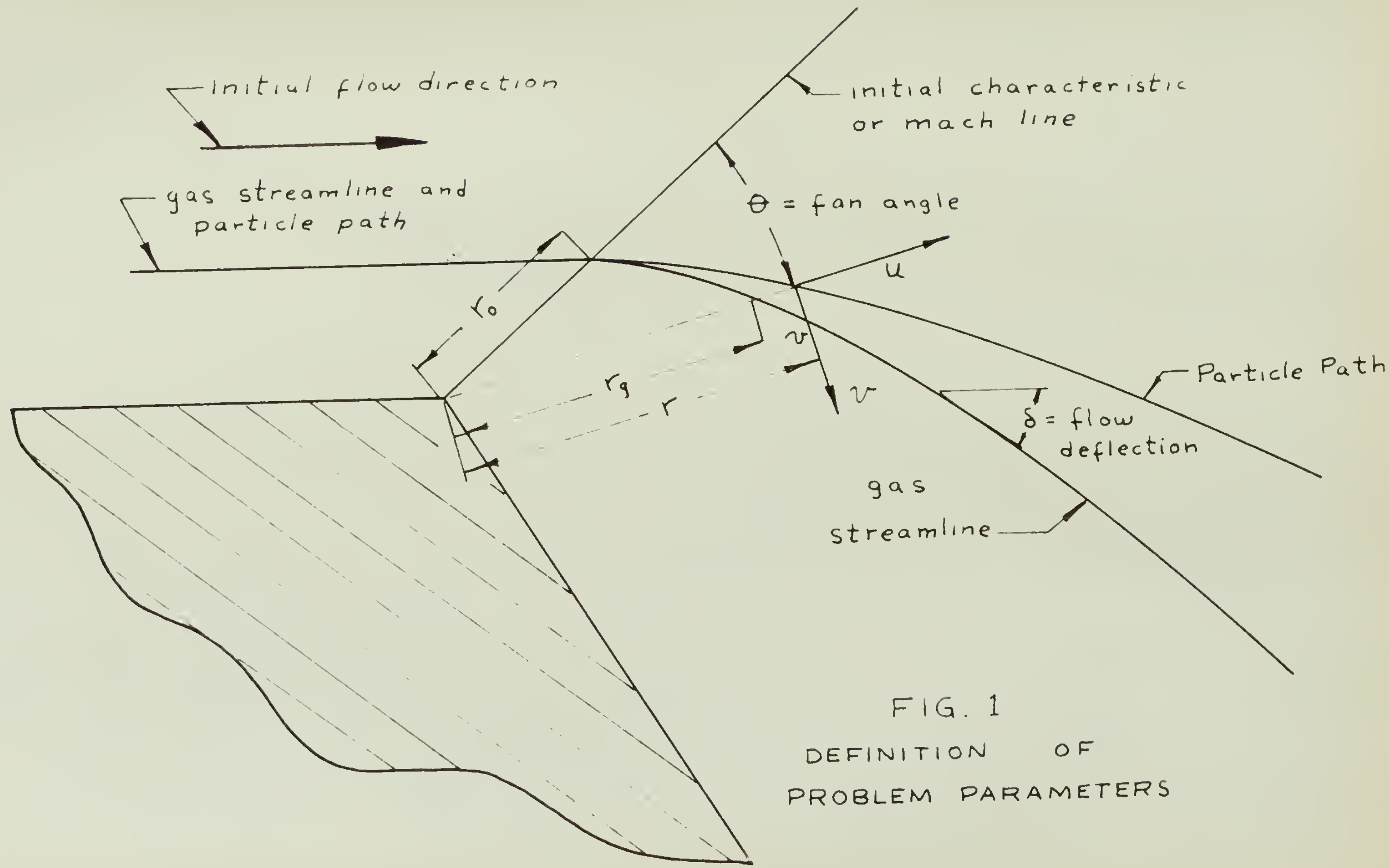

FIG. 1
DEFINITION OF PROBLEM PARAMETERS

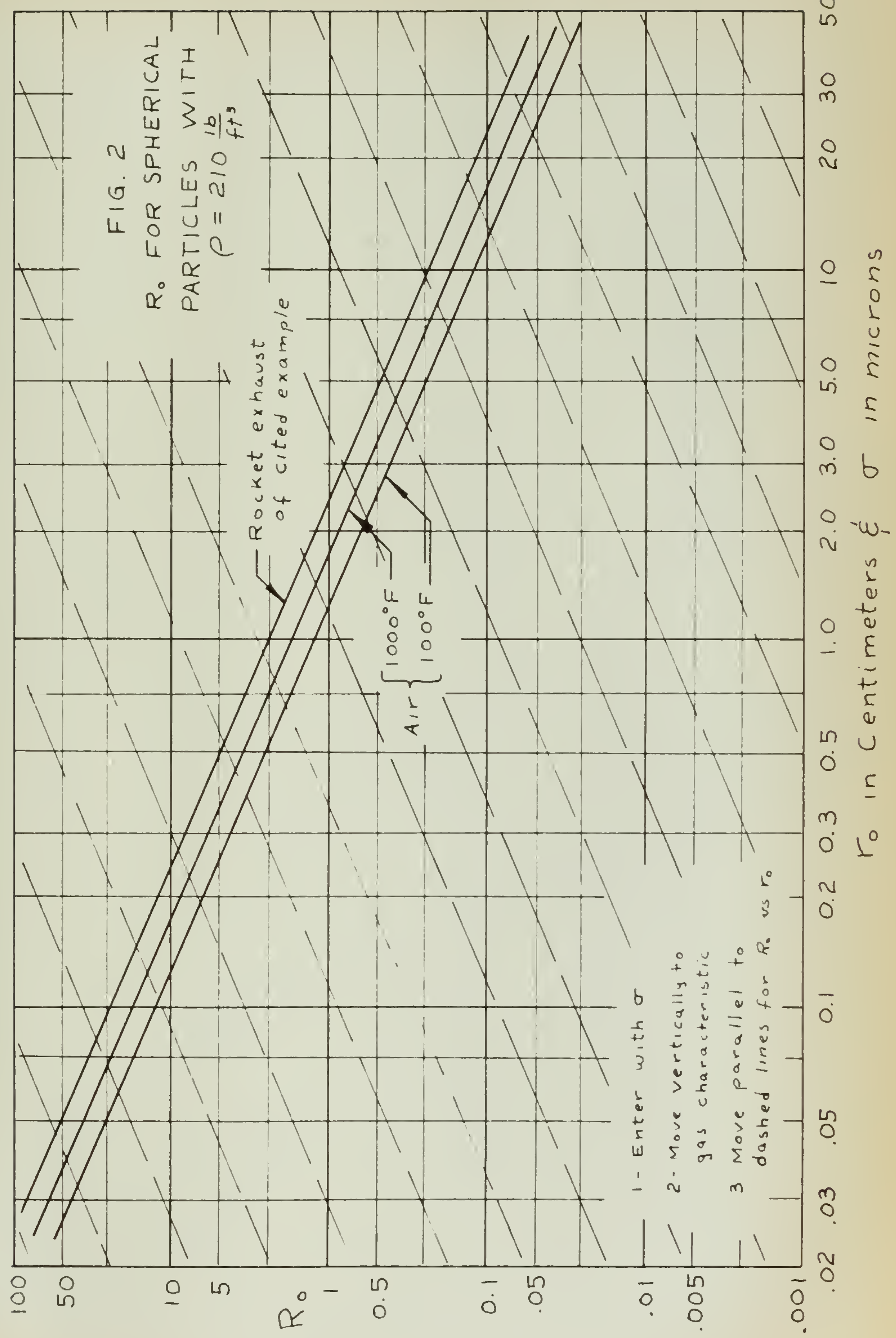
FIG. 2
R_o FOR SPHERICAL PARTICLES WITH $\rho = 210 \frac{lb}{ft^3}$
Rocket exhaust of cited example
Air { 1000°F, 100°F
1 - Enter with σ
2 - Move vertically to gas characteristic
3 Move parallel to dashed lines for R_o vs r_o
R_o
100 50 10 5 1 0.5 0.1 .05 .01 .005 .001
.02 .03 .05 0.1 0.2 0.3 0.5 1.0 2.0 3.0 5.0 10 20 30 50
r_o in Centimeters & σ in microns

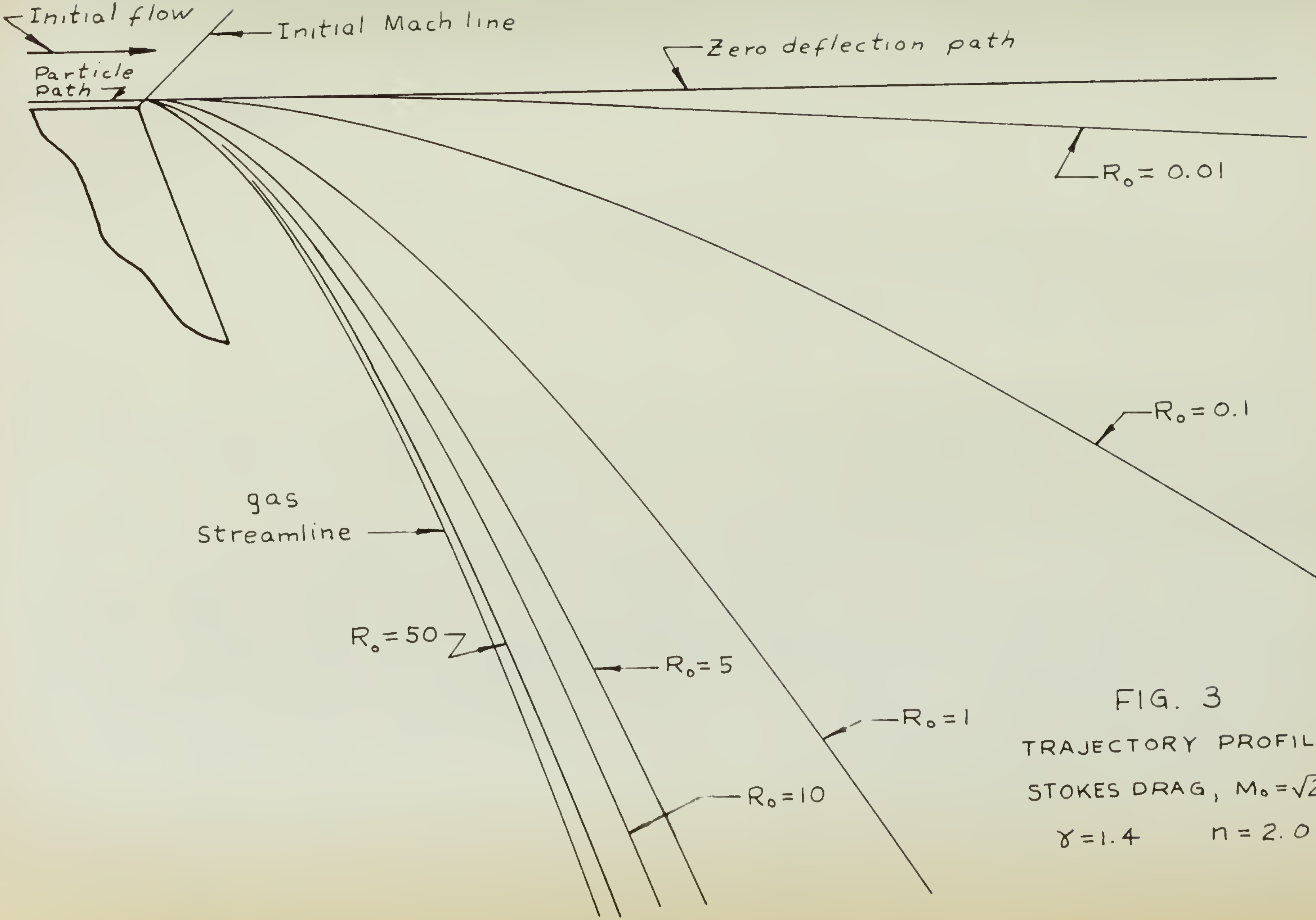

FIG. 3

TRAJECTORY PROFILES

STOKES DRAG, $M_o = \sqrt{2}$

$\gamma = 1.4 \qquad n = 2.0$

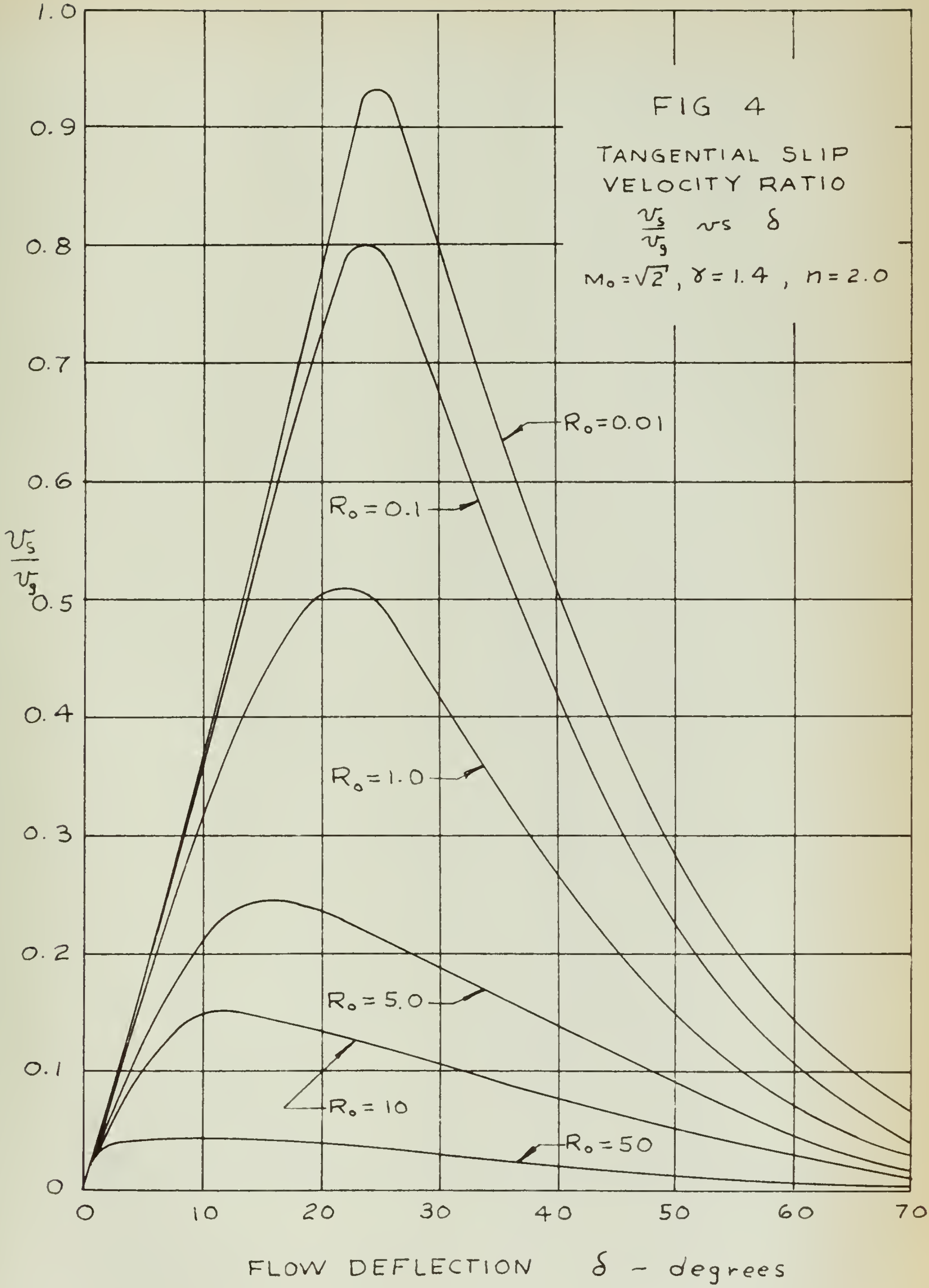
FIG 4
TANGENTIAL SLIP VELOCITY RATIO
$\frac{v_s}{v_g}$ vs δ
$M_o = \sqrt{2}$, $\gamma = 1.4$, $n = 2.0$
$R_o = 0.01$
$R_o = 0.1$
$R_o = 1.0$
$R_o = 5.0$
$R_o = 10$
$R_o = 50$
$\frac{v_s}{v_g}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0
10
20
30
40
50
60
70
FLOW DEFLECTION δ – degrees

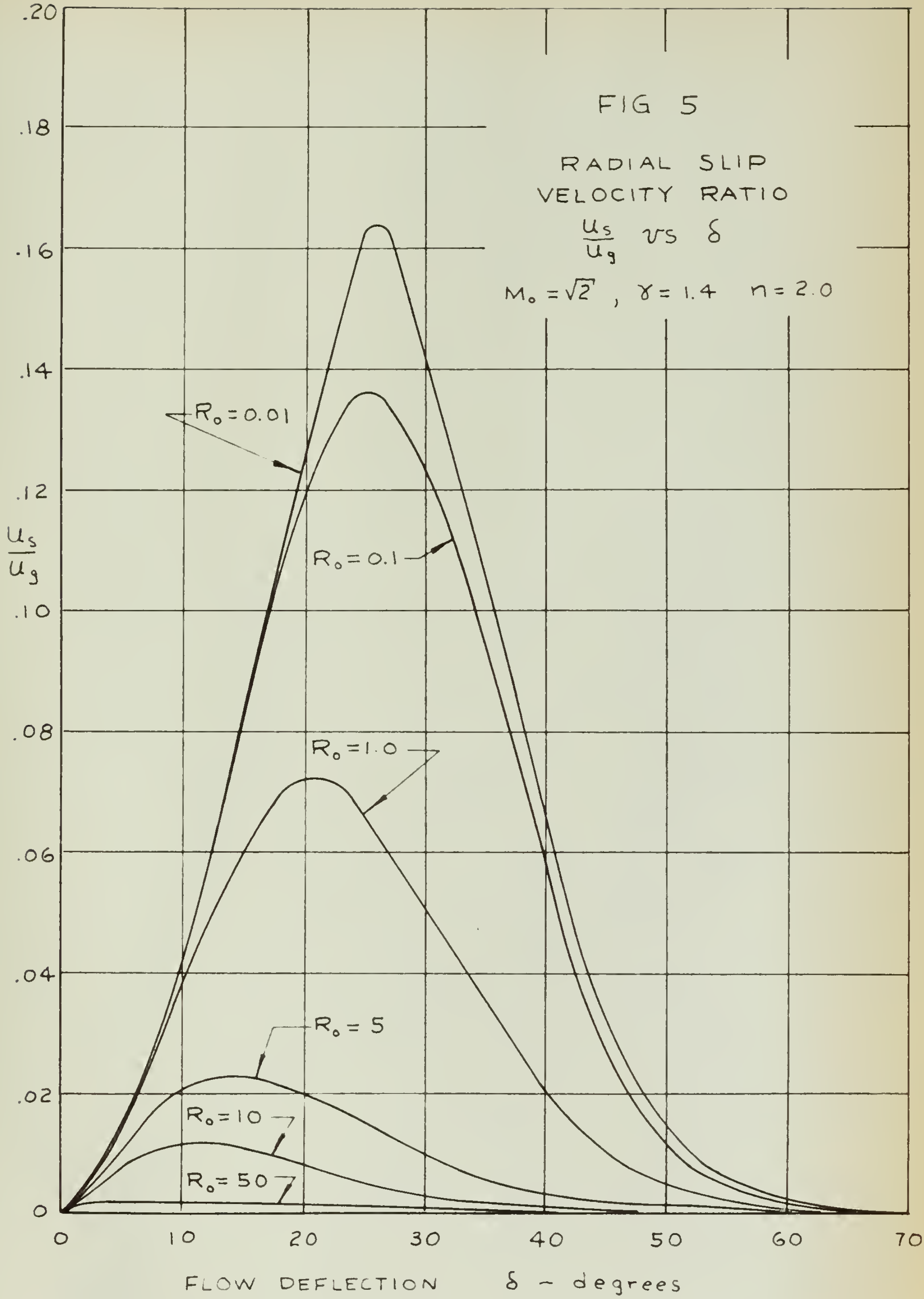
FIG 5
RADIAL SLIP
VELOCITY RATIO
$\frac{u_s}{u_g}$ vs δ
$M_o = \sqrt{2}$, $\gamma = 1.4$ $n = 2.0$
$R_o = 0.01$
$R_o = 0.1$
$R_o = 1.0$
$R_o = 5$
$R_o = 10$
$R_o = 50$
$\frac{u_s}{u_g}$
.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
0
0
10
20
30
40
50
60
70
FLOW DEFLECTION δ ~ degrees

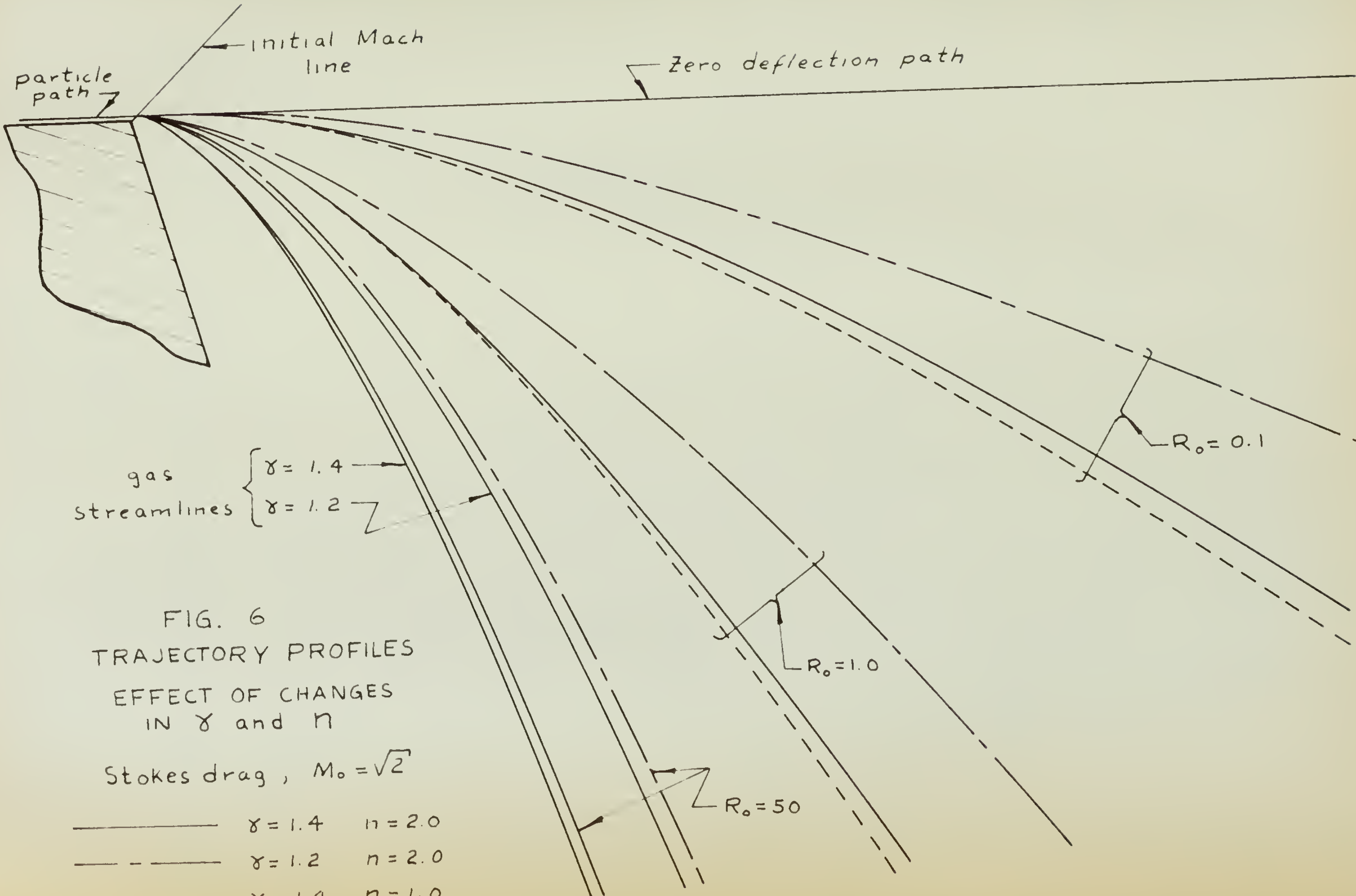

FIG. 6
TRAJECTORY PROFILES
EFFECT OF CHANGES IN γ and n

Stokes drag, $M_o = \sqrt{2}$

————	$\gamma = 1.4$	$n = 2.0$
—— - ——	$\gamma = 1.2$	$n = 2.0$
- - - - - - -	$\gamma = 1.4$	$n = 1.0$

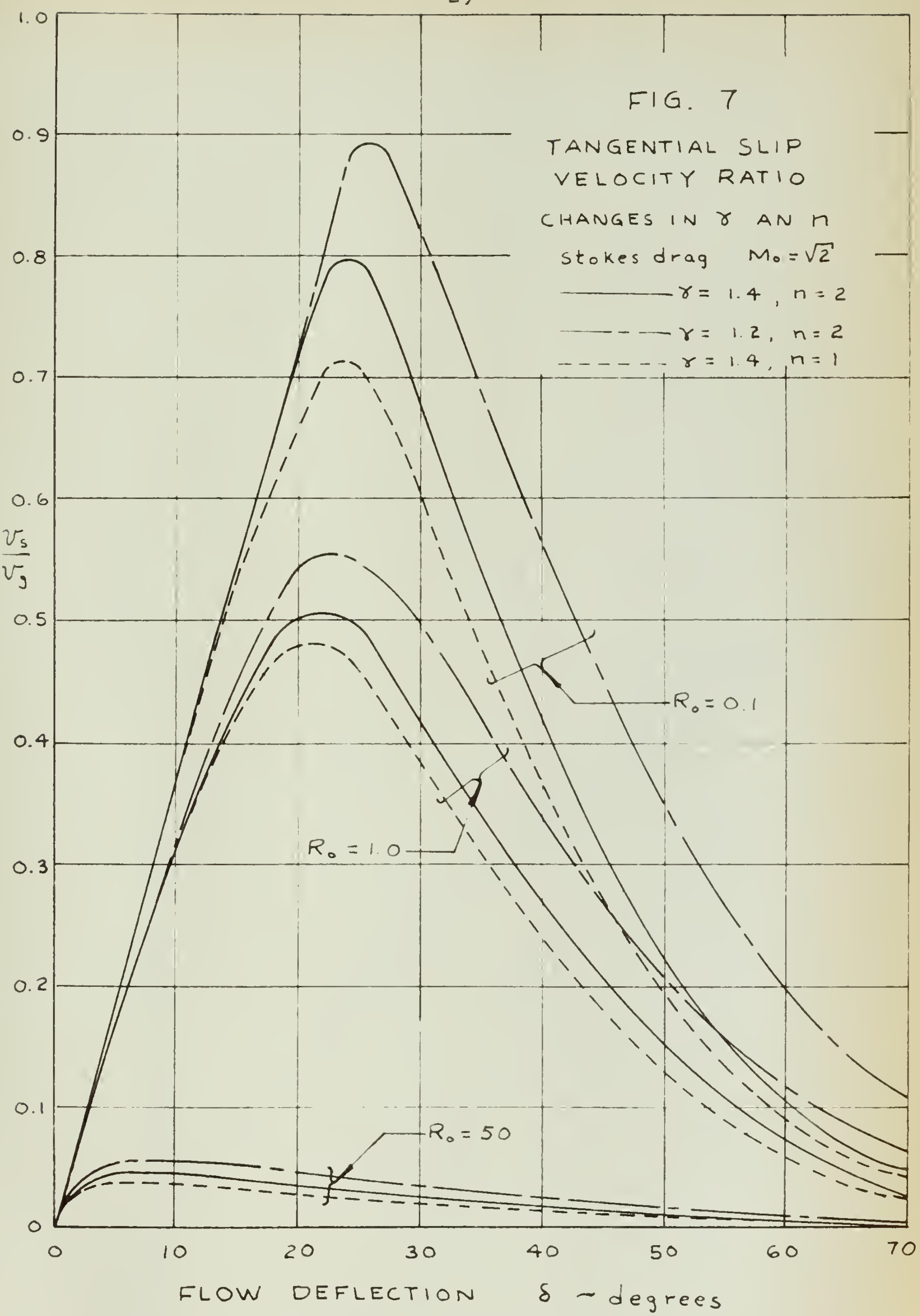
FIG. 7
TANGENTIAL SLIP VELOCITY RATIO
CHANGES IN γ AN n
Stokes drag $M_o = \sqrt{2}$
γ = 1.4, n = 2
γ = 1.2, n = 2
γ = 1.4, n = 1
$R_o = 0.1$
$R_o = 1.0$
$R_o = 50$
$\frac{v_s}{V_g}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
FLOW DEFLECTION δ ~ degrees

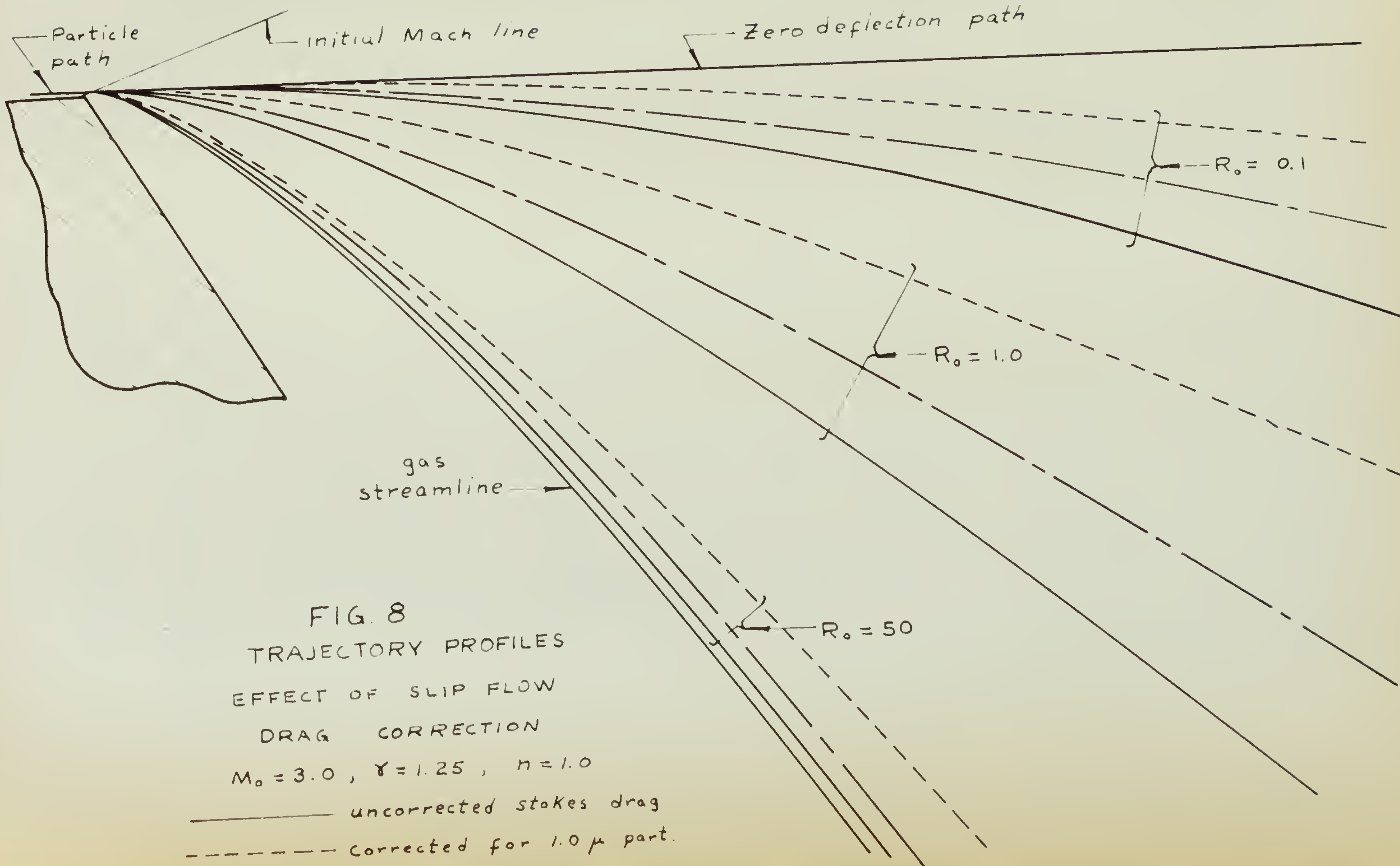

FIG. 8
TRAJECTORY PROFILES
EFFECT OF SLIP FLOW
DRAG CORRECTION
$M_o = 3.0$, $\gamma = 1.25$, $n = 1.0$

——— uncorrected stokes drag
- - - - corrected for 1.0 μ part.
— - — corrected for 4.0 μ part.

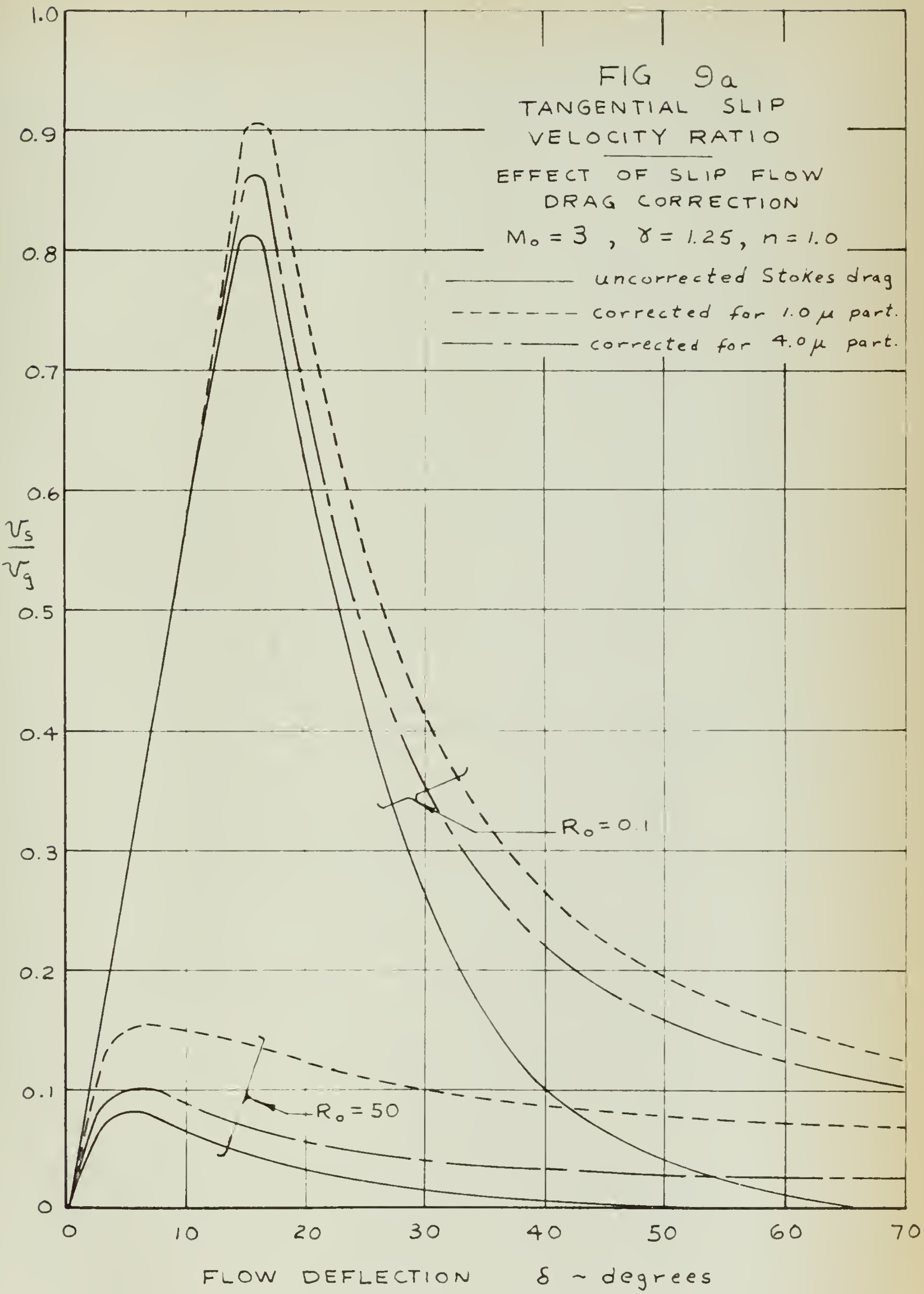
FIG 9a
TANGENTIAL SLIP
VELOCITY RATIO
EFFECT OF SLIP FLOW
DRAG CORRECTION
$M_o = 3$, $\gamma = 1.25$, $n = 1.0$
uncorrected Stokes drag
corrected for 1.0 μ part.
corrected for 4.0 μ part.
$R_o = 0.1$
$R_o = 50$
$\frac{V_s}{V_g}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
FLOW DEFLECTION δ ~ degrees

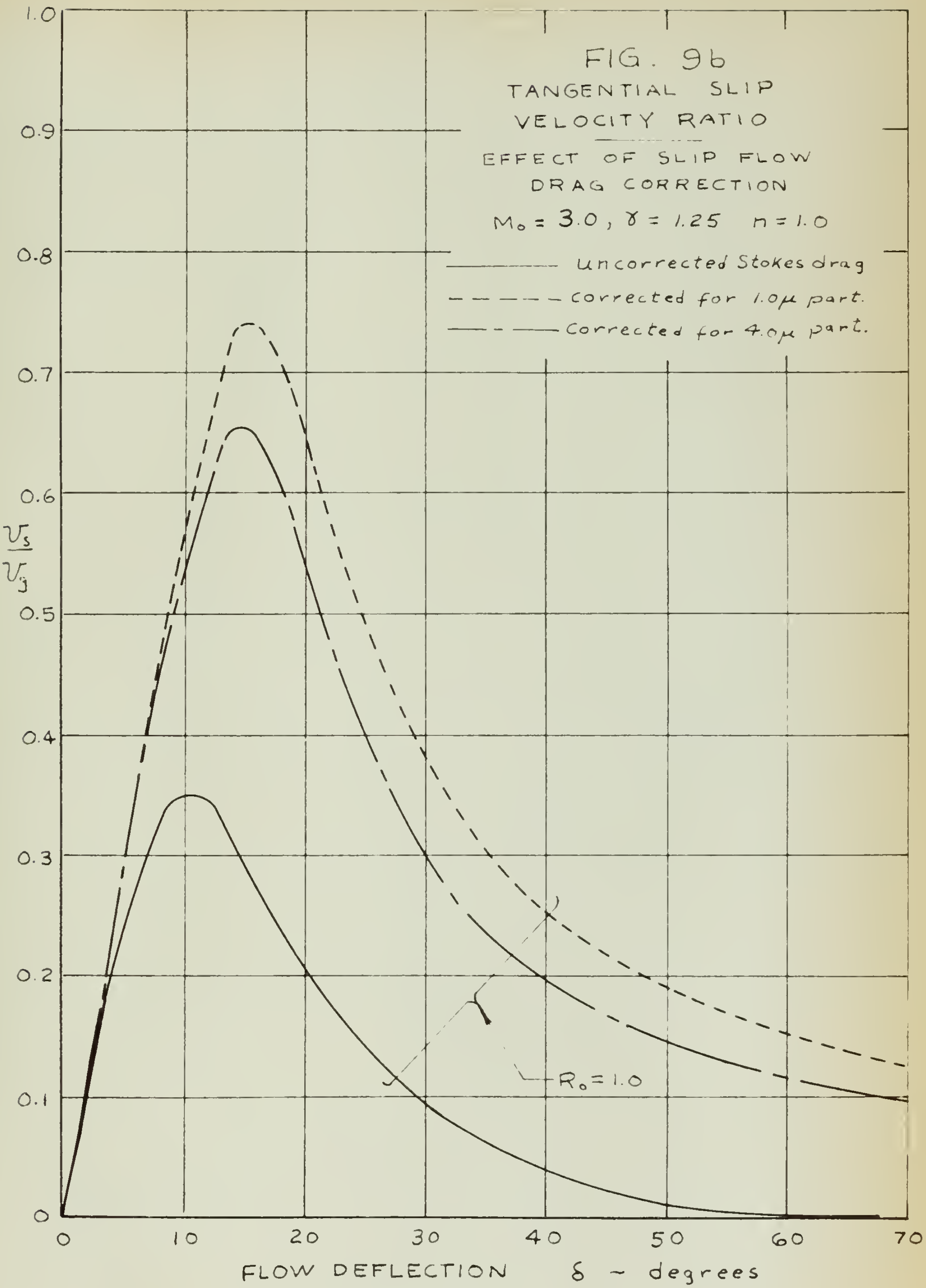
FIG. 9b
TANGENTIAL SLIP
VELOCITY RATIO
EFFECT OF SLIP FLOW
DRAG CORRECTION
$M_o = 3.0$, $\gamma = 1.25$ $n = 1.0$
Uncorrected Stokes drag
Corrected for 1.0μ part.
Corrected for 4.0μ part.
$R_o = 1.0$
$\frac{V_s}{V_j}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
FLOW DEFLECTION δ ~ degrees

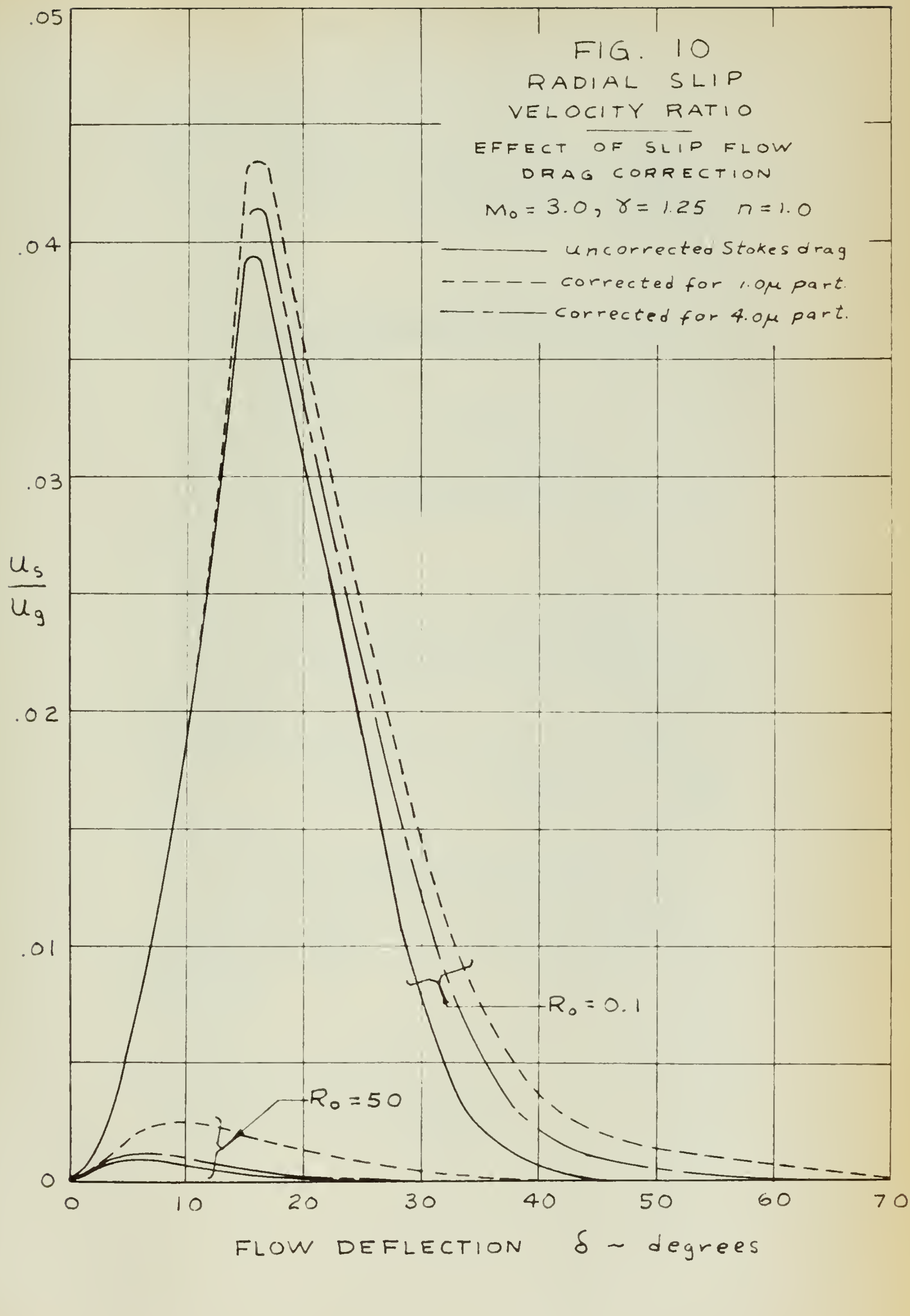
FIG. 10
RADIAL SLIP VELOCITY RATIO
EFFECT OF SLIP FLOW DRAG CORRECTION
$M_o = 3.0$, $\gamma = 1.25$ $n = 1.0$
uncorrected Stokes drag
corrected for 1.0μ part.
corrected for 4.0μ part.
$R_o = 0.1$
$R_o = 50$
$\frac{u_s}{u_g}$
.05
.04
.03
.02
.01
0
0
10
20
30
40
50
60
70
FLOW DEFLECTION δ ~ degrees

APPENDIX A

FORTRAN STATEMENT

```
      DIMENSION Y(10), YDOT(3), C(10), D(10)

      DIMENSION Y(10), YDOT(3), C(10), D(10)
  500 FORMAT (7F10.4)
  510 FORMAT (10H1 GAMMA = E13.6, 13H     MZERO = E13.6,
     19H     N = F5.2, 13H     RZERO = E13.6, 13H     DELTA = E13.6//)
   10 READ INPUT TAPE 5,500, CGAMMA, CMZERO, CN, RZERO,DELT, CMRE, CD
      WRITE OUTPUT TAPE 6, 510, CGAMMA, CMZERO, CN, RZERO, DELT
      Y(1) = 1.0
      Y(2) = 1.0
      Y(3) = 0.0
      T = 0.0
      KAY = 1
      CLAMDA = SQRTF((CGAMMA-1.0)/(CGAMMA+1.0))
      B = (1.0/(CLAMDA**2))-CN+1.0
      ARG = SQRTF (1.0+CLAMDA*CLAMDA*(CMZERO*CMZERO-1.0))
      ALPHA = ACOSF(1.0/ARG)
      C(1) = (1.0/CLAMDA)-CLAMDA
      C(2) = C(1)/CLAMDA
      C(3) = COSF (ALPHA)
      C(4) = CLAMDA*(C(3)**(B-1.0))*RZERO
      C(5) = 2.0/CLAMDA
      C(6) = CMRE / CD
      WRITE OUTPUT TAPE 6,500,CLAMDA,B,ARG,ALPHA
      WRITE OUTPUT TAPE 6,500,C(1),C(2),C(3),C(4),C(5), C(6)
      CALL DEQ(K,3,T,Y,YDOT,.01,5.0E-7)
      GO TO (100,200,200,300),K
  100 D(1) = CLAMDA*T+ALPHA
      D(2) = TANF(D(1))
      D(3) = COSF(D(1))
      D(4) = (1.0/(D(3)**B))
      D(5) = 1.0/(1.0+C(6) * ((C(3)/D(3)) **(B-1.0)))
      YDOT(1) = Y(3)
      YDOT(3) = (C(5)) *Y(3) *D(2) - C(2)*Y(1)*D(2)*D(2)
     1-(C(4)*Y(3)/((Y(1)**3)*Y(2)*Y(2)))*D(4) * D(5)
      YDOT(2) = -C(1)*Y(2)*D(2)+C(4)*D(4)*(1.0/((Y(1)**3)*Y(2))
     1-1.0/(Y(1)*Y(1))) *D(5)
      IF (KAY-2) 150, 160, 160
  160 CALL DEQ2
  150 WRITE OUTPUT TAPE 6, 500, D(1), D(2), D(3), D(4),D(5)
      WRITE OUTPUT TAPE 6, 600
  600 FORMAT (//6H THETA,9H        X, 14H             Y,
     114H              Z, 16H              V/VS,14H          U/US,
     214H            R/RC, 13H            MS, 13H           RE//)
      KAY = 2
      GO TO 160
  200 Y(4) = Y(1)*Y(2)
      Y(5) = Y(4)*(1.0-CLAMDA*YDOT(1)/(Y(1)*D(2)))
      Y(6) = ((C(3)/D(3))**(C(2)+1.0))/Y(1)
      Y(9) = ((1.0-Y(4))**2)+(((1.0-Y(5))*D(2)/CLAMDA)**2)
      Y(7) = SQRTF(Y(9))
      Y(8) = Y(7)*((D(3)/C(3))**(B-1.0))
      WRITE OUTPUT TAPE 6, 610, T, (Y(JACK),JACK = 1,8)
  610 FORMAT(/F7.3,8E14.6)
  205 IF(T-3.0)210,210,220
  210 CALL DEQ1
      GO TO 205
  220 GO TO 10
      DIMENSION Y(10), YDOT(3), C(10), D(10)

  300 WRITE OUTPUT TAPE 6, 620
  620 FORMAT (22H ERROR RETURN FROM DEQ)
      GO TO 10
      END(1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)
```